

Compact stellar objects with multiple neck optical geometries

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To appear in Annalen der Physik

Abstract. Ultracompact stellar models with a two-zone uniform density equation of state are considered. They are shown to provide neat examples of optical geometries exhibiting double necks, implying that the gravitational wave potential has a double well structure.

Keywords: general relativity, relativistic stars, gravitational waves

PACS: 04.30.Db, 04.40.Dg

1 Introduction

The *static spherically symmetric* (SSS) relativistic stellar models have been the subject of extensive studies in the past (see for example [1] for a review of exact perfect fluid SSS solutions of the Einstein equations up to 1994). De Felice [2] first discussed the possibility of trapping of massless particles in the interior of compact stellar objects. Abramowicz and coworkers [3] considered the optical geometry of uniform density models. They found that the optical geometry (for geodesics in the equatorial plane) of the stellar interior was identical to a standard 2-sphere. They also showed that the optical geometry of the uniform density models with $R < 3M$ had a neck (at $R = 3M$) and a corresponding bulge in the interior. The bulge is associated with stable spatially closed orbits of massless (or ultrarelativistic) particles and waves propagating at the speed of light (in the geometric optics approximation). This provided a nice visual illustration of the phenomenon of trapping of waves and matter moving at relativistic velocities inside compact objects. Moreover, it allowed a simple derivation of the eigenfrequencies of the w-modes of trapped gravitational waves.

Despite the relative simplicity of the field equations for static spherically symmetric perfect fluid models, the full picture of the structure of the gravitational field of such objects is only now beginning to emerge. Following the powerful Nilsson-Uggla formulation of the SSS field equations [4], a novel feature which was recently uncovered is the existence of optical geometries with multiple necks [5]. One might naively expect that such behavior would only occur for rather strange or unphysical equations of state. However, as shown in [5], multiple necks do in fact appear for the innocent looking Zel'dovich stiff matter equation of state with an added constant $p = \rho - \rho_s$.

We are using a subscript “s” to denote evaluation at zero pressure (*i.e.* at the stellar surface). In fact if n is the number of the necks, then $n \rightarrow \infty$ for stellar models with that equation of state in the limit of infinite central density, $\rho_c \rightarrow \infty$, where the subscript “c” is used for evaluation at the center of the stellar object. Actually, most of these models correspond to unstable equilibria and so would not be realizable as stable stellar objects. However, unstable equilibria can play a role as intermediate states in gravitational collapse situations [6].

The Zel’dovich equation of state has the advantage of being causal in the sense that the adiabatic speed of sound equals the speed of light, $v_{\text{sound}} := \sqrt{dp/d\rho} = 1$. However, from the point of view of visualizing the multiple neck optical geometry, it is not so well suited because of the rather small amplitude of the necks for that equation of state. Therefore, to obtain better visual displays we have used double layer uniform density models to exhibit double neck optical geometries. Such models were used by Lindblom [7] as prototypes to discuss phase transitions in relativistic stellar objects.

2 Some multiple neck optical geometries

The metric for a SSS spacetime can be written in the Schwarzschild radial gauge as

$$ds^2 = -e^{2\nu} dt^2 + e^{2\lambda} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where ν and λ are functions of r . When studying null geodesics, it is often convenient to introduce the *optical geometry* of the spacetime, defined by

$$d\tilde{s}^2 = e^{-2\nu} ds^2 = -dt^2 + e^{2(\lambda-\nu)} dr^2 + e^{-2\nu} r^2 d\phi^2. \quad (2)$$

Since this metric is conformally related to the physical metric, the spacetime and its optical geometry have the same null geodesics. Due to the symmetries of the spacetime we may, without loss of generality, consider only the spacelike slice given by $t = 0$ and $\theta = \pi/2$. Introducing the tortoise radial variable defined by $dr_* = e^{\lambda-\nu} dr$, the metric on this slice may be written

$$dl^2 = dr_*^2 + \tilde{r}^2 d\phi^2, \quad (3)$$

where $\tilde{r} = e^{-\nu} r$. In order to gain intuitive understanding of this geometry it is useful to embed it in 3-dimensional Euclidean space. Standard embedding techniques (*c.f.* *e.g.* [8]) yield the differential equation for the embedded surface

$$\tilde{r} = e^{-\nu} r, \quad h' = e^{-\nu} \sqrt{e^{2\lambda} - (1 - r\nu')^2}, \quad (4)$$

where \tilde{r} and h are the radius and height coordinates in a standard cylindrical coordinate system and a prime denotes differentiation with respect to r .

As shown by Chandrasekhar and Ferrari [9] the axial modes of nonradial metric perturbations of SSS perfect fluids are (to first order) not coupled to a perturbation of the matter. Such metric perturbations may therefore be interpreted as pure gravitational waves. By separation of variables the equations governing these modes can be reduced to the 1-dimensional Schrödinger equation

$$-\frac{d^2}{dr_*^2} Z_l + (V_l + V) Z_l = \sigma^2 Z_l \quad (5)$$

where the total potential $V_l + V$ is dominated by $V_l = l(l+1)\tilde{r}^{-2}$, $l = 2, 3, \dots$ which is simply the effective potential for null geodesics with angular momentum $L^2 = l(l+1)$. This shows that the optical geometry is closely related to the trapped modes of gravitational radiation since the maxima and minima of V_l clearly correspond to the necks and bulges of the optical geometry respectively. We turn now to the analysis of the optical geometry of the double layer uniform density models. The equation of state is parametrized by the constant densities ρ_+ and ρ_- of the interior and exterior zone respectively, as well as the transition pressure p_t where the density is discontinuous. Given these parameters, a stellar model is obtained by specifying the central pressure p_c . Below we display the optical geometry and the axial mode potentials for three stars of this type. The parameter values are chosen to obtain three distinct examples of optical geometries with pronounced double necks, corresponding to two deep potential wells. As the wells are fairly rectangular-shaped with a certain width L and height V_0 , one can make a crude estimate of the number of trapped modes by taking Z_l to be the wave function of an infinite square well, with the same width L , and count the number of states with energy less than V_0 . The number of trapped modes is then approximately given by $\sqrt{V_0}L/\pi$.

The models considered here were chosen to have the same value of the tenuity $\alpha = R/M = 2.2749$, and the overall scale was fixed by setting the total mass equal to unity (in units such that $G_{ab} = T_{ab}$). Below the optical geometries are displayed next to the potentials relevant for the axial modes with $l = 2$, the solid line indicating the total potential ($V + V_l$), the dashed line showing V_l and the dash-dotted line showing V . In the optical geometry the two shades of gray indicate the different density layers, and the white part corresponds to the exterior Schwarzschild geometry. The parameter values, given in units of the appropriate power of the total mass, are given in the respective captions. We may remark however that none of these models are likely to be stable, but that we have found models with less pronounced double necks that are probably stable.

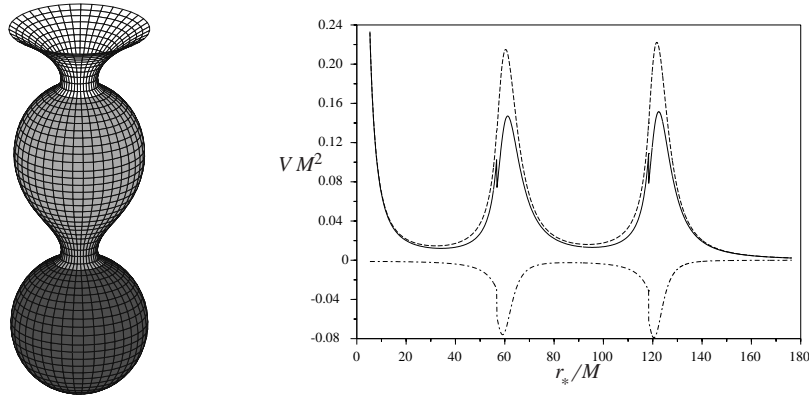


Fig. 1 The “peanut” geometry and the corresponding potentials. The parameter values are $\rho_+ = 2.512 \times 10^3$, $\rho_- = 0.5024$, $p_t = 35.89$, and $p_c = 5.778 \times 10^4$.

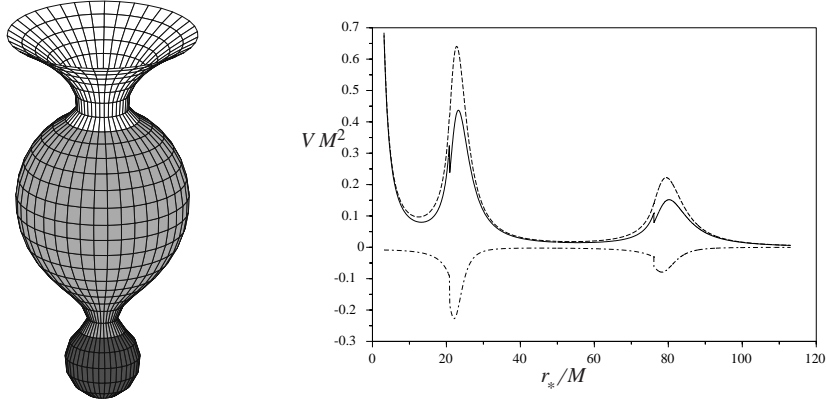


Fig. 2 The “bigup” geometry and the corresponding potential. The parameter values are $\rho_+ = 5.325 \times 10^3$, $\rho_- = 0.5047$, $p_t = 29.58$, and $p_c = 5.325 \times 10^4$.

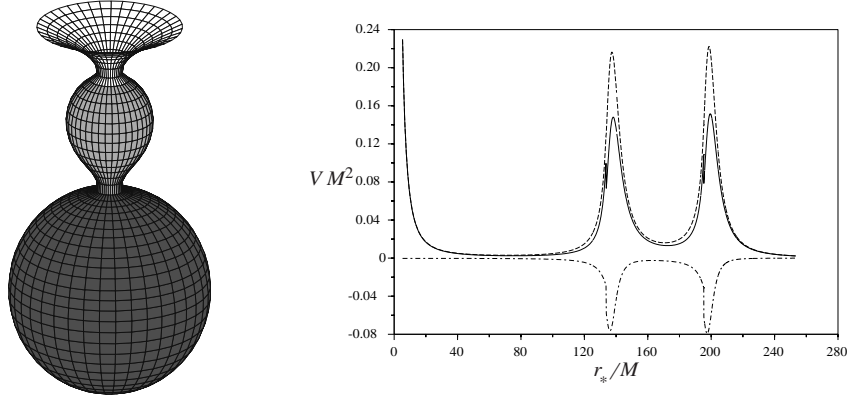


Fig. 3 The “bigdown” geometry and the corresponding potential. The parameter values are $\rho_+ = 2.569 \times 10^3$, $\rho_- = 0.5024$, $p_t = 36.69$, and $p_c = 2.954 \times 10^5$.

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